

On the some properties of circulant matrix with third order linear recurrent sequence

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Abstract

In this paper, firstly, we give the some fundamental properties of Van Der Laan numbers. After, we define the circulant matrices $C(Z)$ which entries is third order linear recurrent sequence. In addition, we compute eigenvalues, spectral norm and determinant of this matrix. Consequently, by using properties of this sequence, we obtain the eigenvalues, norms and determinants of circulant matrices with Cordonnier, Perrin and Van Der Laan numbers.

Keywords: Third order linear recurrent sequence, Cordonnier numbers, Perrin numbers, Van Der Laan numbers, circulant matrix, norm, determinant.

1 Introduction

Shannon, Horadam and Anderson [4] defined respectively Cordonnier sequence $\{P_n\}_{n \in \mathbb{N}}$ as

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1, \quad (1)$$

Perrin sequence $\{Q_n\}_{n \in \mathbb{N}}$ as

$$Q_{n+3} = Q_{n+1} + Q_n, \quad Q_0 = 3, Q_1 = 0, Q_2 = 2, \quad (2)$$

and Van Der Laan sequence $\{R_n\}_{n \in \mathbb{N}}$ as

$$R_{n+3} = R_{n+1} + R_n, \quad R_0 = 0, R_1 = 1, R_2 = 0. \quad (3)$$

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The fact that the Cordonnier numbers and Perrin numbers are a linear combination of α^n , β^n and γ^n , that is,

$$P_n = \frac{\alpha^{n+4}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+4}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+4}}{(\gamma - \alpha)(\gamma - \beta)}, \quad (4)$$

$$Q_n = \alpha^n + \beta^n + \gamma^n. \quad (5)$$

Hence, the relations are hold

$$\alpha + \beta + \gamma = 0, \alpha\beta\gamma = 1, \alpha\beta + \beta\gamma + \alpha\gamma = -1, \quad (6)$$

where α , β and γ are roots of equations (1), (2), (3).

Elia [12], gave third order linear recurrent sequence $\{T_0, T_1, T_2, \dots\}$ defined by the recurrence

$$T_{n+3} = pT_{n+2} + qT_{n+1} + rT_n, \quad T_0 = a, T_1 = b, T_2 = c. \quad (7)$$

Also, he studied Tribonacci cubic form and solve the integer representation problem for the Tribonacci cubic form.

Recently, it has been widely studied the some properties of the circulant matrices with special numbers. For instance, in [6], they defined the generalized k -Horadam numbers and computed the spectral norm, eigenvalues and the determinant of circulant matrix with this numbers. Solak [7] defined the $n \times n$ circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\text{mod}(j-i, n))}$ and $b_{ij} \equiv L_{(\text{mod}(j-i, n))}$. Additionally, he investigated the upper and lower bounds of the matrices A and B , respectively. In [5], Ipek obtained the spectral norms of circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\text{mod}(j-i, n))}$ and $b_{ij} \equiv L_{(\text{mod}(j-i, n))}$. Shen and Cen, in [8, 11], have found upper and lower bounds for the spectral norms of r -circulant matrices and obtained some bounds for the spectral norms of Kronecker and Hadamard products of these matrices. Also, they gave the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. In [3], it has been studied the norms, eigenvalues and determinants of some matrices related to different numbers. Yazlik [9] obtained upper and lower bounds for the spectral norm of an r -circulant matrices $H = C_r(H_{k,0}, H_{k,1}, H_{k,2}, \dots, H_{k,n-1})$ whose entries are the generalized k -Horadam numbers. Additionally, he find new formulas to calculate the eigenvalues and determinant of the matrix H . In [10], they defined the circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers and computed the determinants and inverses of these matrices.

In the light of the above studies, in here, we present some properties of the Van Der Laan sequence as Binet formula, sum. After, we find eigenvalues, spectral norm and determinant of circulant matrix with the third order sequence. Finally, we give eigenvalues, norms and determinants of circulant matrices with Cordonnier, Perrin and Van Der Laan numbers via properties of circulant matrix with this third order sequence.

Now, we give some preliminaries about circulant matrix and the spectral norm of a matrix.

The circulant matrix $C = [c_{ij}] \in M_{n,n}(\mathbb{C})$ is defined by the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i \\ c_{n+j-i}, & j < i \end{cases}$$

The spectral norm of A , for a matrix $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$, is given by

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^*A)},$$

where A^* is the conjugate transpose of matrix A .

Lemma 1 [2] *Let $A = \text{circ}(a_0, a_1, \dots, a_{n-1})$ be a $n \times n$ circulant matrix. Then we have*

$$\lambda_j(A) = \sum_{k=0}^{n-1} a_k w^{-jk},$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

Lemma 2 [1] *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, \dots, |\lambda_n|^2$, where A^* is the conjugate transpose of the matrix A .*

2 Circulant Matrices with Third Order Linear Sequences

Firstly, since some of results of this paper concern about the spectral norm, eigenvalues and determinant of the circulant matrix entried by the Van Der Laan numbers, we need to introduce some properties of this sequence.

The Binet formula of Van Der Laan sequence, for every $n \in \mathbb{N}$, is obtained as

$$R_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}, \quad (8)$$

where α, β and γ are roots of equation (3).

Also, for $n \geq 1$, we have

$$\sum_{k=0}^n R_k = R_{n+5} - 1,$$

$$\sum_{k=0}^n R_k^2 = R_{n+2}^2 - R_{n-1}^2 - R_{n-3}^2 + 1.$$

If we take $p = 0, q = 1, r = 1$ in (7), we obtain as

$$Z_{n+3} = Z_{n+1} + Z_n, \quad (9)$$

where initial conditions $Z_0 = a$, $Z_1 = b$, $Z_2 = c$. Also, the Binet formula of Z_n sequence, for every $n \in \mathbb{N}$, is obtained as

$$Z_n = \frac{(\alpha^2 - 1)a + \alpha b + c}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n + \frac{(\beta^2 - 1)a + \beta b + c}{(\beta - \alpha)(\beta - \gamma)} \beta^n + \frac{(\gamma^2 - 1)a + \gamma b + c}{(\gamma - \alpha)(\gamma - \beta)} \gamma^n, \quad (10)$$

where α , β and γ are roots of equation $x^3 - x - 1 = 0$.

Proposition 3 For $n \geq 1$, the following relations are hold:

$$\sum_{k=0}^n Z_k = Z_{n+5} - Z_4,$$

$$\sum_{k=0}^n Z_k^2 = Z_{n+2}^2 - Z_{n-1}^2 - Z_{n-3}^2 + T$$

where $T = 2a(a - c) - (b + c)^2$.

We formulate eigenvalues, spectral norms and determinants of the circulant matrices with the third order linear sequences. In order to do that, we can define the *circulant matrix* as follows.

Definition 4 $n \times n$ circulant matrix with third order linear sequence entries is defined by

$$C(Z) = \begin{bmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_{n-1} \\ Z_{n-1} & Z_0 & Z_1 & \cdots & Z_{n-2} \\ Z_{n-2} & Z_{n-1} & Z_0 & \cdots & Z_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_1 & Z_2 & Z_3 & \cdots & Z_0 \end{bmatrix}. \quad (11)$$

The following theorem gives us the eigenvalues of circulant matrix with third order linear recurrent sequence.

Theorem 5 Let $C(Z) = \text{circ}(Z_0, Z_1, \dots, Z_{n-1})$ be circulant matrix. Then the eigenvalues of $C(Z)$ are

$$\lambda_j(C(Z)) = \frac{Z_n - a + (Z_{n+1} - b)w^{-j} + (Z_{n-1} - c + a)w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

Proof. By taking $X = (\alpha - \beta)(\alpha - \gamma)$, $Y = (\beta - \alpha)(\beta - \gamma)$, $W = (\gamma - \alpha)(\gamma - \beta)$, from

Lemma 1, we have

$$\begin{aligned}
\lambda_j(C(Z)) &= \sum_{k=0}^{n-1} Z_k w^{-jk} \\
&= \sum_{k=0}^{n-1} \left(\frac{(\alpha^2-1)a+\alpha b+c}{X} \alpha^k + \frac{(\beta^2-1)a+\beta b+c}{Y} \beta^k \right. \\
&\quad \left. + \frac{(\gamma^2-1)a+\gamma b+c}{W} \gamma^k \right) w^{-jk} \\
&= \frac{(\alpha^2-1)a+\alpha b+c}{X} \left(\frac{(\alpha w^{-j})^n - 1}{\alpha w^{-j} - 1} \right) + \frac{(\beta^2-1)a+\beta b+c}{Y} \left(\frac{(\beta w^{-j})^n - 1}{\beta w^{-j} - 1} \right) \\
&\quad + \frac{(\gamma^2-1)a+\gamma b+c}{W} \left(\frac{(\gamma w^{-j})^n - 1}{\gamma w^{-j} - 1} \right) \\
&= \frac{(\alpha^n-1)[(\alpha^2-1)a+\alpha b+c]}{X(\alpha w^{-j}-1)} \\
&\quad + \frac{(\beta^n-1)[(\beta^2-1)a+\beta b+c]}{Y(\beta w^{-j}-1)} + \frac{(\gamma^n-1)[(\gamma^2-1)a+\gamma b+c]}{W(\gamma w^{-j}-1)}.
\end{aligned}$$

By considering (6) and (10), we obtain

$$\lambda_j(C(Z)) = \frac{Z_n - a + (Z_{n+1} - b)w^{-j} + (Z_{n-1} - c + a)w^{-2j}}{w^{-3j} + w^{-2j} - 1}$$

which is desired. ■

By using Theorem 5, we obtain eigenvalues of circulant matrices with Cordonnier, Perrin and Van Der Laan sequences as follows.

Corollary 6 *i) Let $C(P) = \text{circ}(P_0, P_1, \dots, P_{n-1})$ be circulant matrix. If we take $a = b = c = 1$ in Theorem 5, then the eigenvalues of $C(P)$ are*

$$\lambda_j(C(P)) = \frac{P_n - 1 + (P_{n+1} - 1)w^{-j} + P_{n-1}w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where P_n is the n th Cordonnier number and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

ii) Let $C(Q)$ be circulant matrix with Perrin numbers. If we take $a = 3, b = 0, c = 2$, the eigenvalues of $C(Q)$ are

$$\lambda_j(C(Q)) = \frac{Q_n - 3 + Q_{n+1}w^{-j} + (Q_{n-1} + 1)w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where Q_n is the n th Perrin number and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

iii) Let $C(R) = \text{circ}(R_0, R_1, \dots, R_{n-1})$ be circulant matrix. If we take $a = c = 0, b = 1$ in theorem 5 the eigenvalues of $C(R)$ are

$$\lambda_j(C(R)) = \frac{R_n + (R_{n+1} - 1)w^{-j} + R_{n-1}w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where R_n is the n th Van Der Laan number and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

Because matrix $C(Z)$ is normal matrix, we can write $\lambda_j (C(Z)C(Z)^*) = |\lambda_j (C(Z))|^2$. Also, since the matrices $C(P)$, $C(Q)$ and $C(R)$ are normal matrices, we can write $\lambda_j (C(P)C(P)^*) = |\lambda_j (C(P))|^2$, $\lambda_j (C(Q)C(Q)^*) = |\lambda_j (C(Q))|^2$ and $\lambda_j (C(R)C(R)^*) = |\lambda_j (C(R))|^2$. Then, we have the following theorem and corollary that deal with spectral norm.

Theorem 7 *Let $C(Z)$ be an $n \times n$ circulant matrix with the Z_n entries. Then, we have*

$$\|C(Z)\|_2 = Z_{n+4} - Z_4.$$

Proof. From Lemma 2, we can write

$$\|C(Z)\|_2 = \sqrt{\left(\max_{0 \leq j \leq n-1} |\lambda_j (C(Z))|^2\right)}.$$

In this last equality, for $j = 0$, λ_0 becomes the maximum eigenvalue. Thus, $\|C(Z)\|_2 = |\lambda_0 (C(Z))|$. Also, from the Theorem 5, we clearly obtain

$$\|C(Z)\|_2 = Z_{n+4} - Z_4.$$

Hence proof is completed. ■

For special cases of a, b, c in Theorem 7, we obtain following corollary for spectral norms of $C(P)$, $C(Q)$ and $C(R)$.

Corollary 8 *i) Let $C(P)$ be an $n \times n$ circulant matrix with the Cordonnier numbers entries. If we take $a = b = c = 1$ in (9), we obtain Cordonnier sequence. And so, from Theorem 7, we obtain*

$$\|C(P)\|_2 = P_{n+4} - 2$$

where P_n is n th Cordonnier number.

ii) Similarly to i), if we take $a = 3, b = 0, c = 2$ in (9), we obtain for $C(Q)$ matrix

$$\|C(Q)\|_2 = Q_{n+4} - 2.$$

where Q_n is n th Perrin number.

iii) Similarly, if we take $a = c = 0, b = 1$ in (9), we have

$$\|C(R)\|_2 = R_{n+4} - 1.$$

where R_n is n th Van Der Laan number.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Since row norm and column norm of the A matrix are $\|A\|_\infty = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}| \right)$ and $\|A\|_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^n |a_{ij}| \right)$, it can be easily seen that $\|C(Z)\|_2 = \|C(Z)\|_1 = \|C(Z)\|_\infty$. Also, $\|C(P)\|_2 = \|C(P)\|_1 = \|C(P)\|_\infty$, $\|C(Q)\|_2 = \|C(Q)\|_1 = \|C(Q)\|_\infty$ and $\|C(R)\|_2 = \|C(R)\|_1 = \|C(R)\|_\infty$.

The following theorem gives us the determinant of $C(Z)$.

Theorem 9 *The determinant of the matrix $C(Z) = \text{circ}(Z_1, Z_2, \dots, Z_n)$ is written by*

$$\det(C(Z)) = \frac{(Z_n - a)^n (1 - K^n - L^n + K^n L^n)}{(-1)^n (Q_{-n} - Q_n)},$$

$$\text{where } K = \frac{b - Z_{n+1} - \sqrt{(Z_{n+1} - b)^2 - 4(Z_n - a)(Z_{n-1} - c + a)}}{2(Z_n - a)},$$

$$L = \frac{b - Z_{n+1} + \sqrt{(Z_{n+1} - b)^2 - 4(Z_n - a)(Z_{n-1} - c + a)}}{2(Z_n - a)}.$$

Proof. From Theorem 5, we have

$$\begin{aligned} \det(C(Z)) &= \prod_{j=0}^{n-1} \lambda_j(C(Z)) \\ &= \prod_{j=0}^{n-1} \frac{Z_n - a + (Z_{n+1} - b)w^{-j} + (Z_{n-1} - c + a)w^{-2j}}{w^{-3j} + w^{-2j} - 1}. \end{aligned}$$

By considering the equality

$$\prod_{k=0}^{n-1} (x - yw^{-k} + zw^{-2k}) = x^n \left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x} \right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x} \right)^n + \left(\frac{z}{x} \right)^n \right),$$

we have

$$\det(C(Z)) = \frac{(Z_n - a)^n (1 - K^n - L^n + K^n L^n)}{(-1)^n (Q_{-n} - Q_n)}.$$

Hence proof is completed. ■

By using in Theorem 9, we obtain determinants of circulant matrices with Cordonnier, Perrin and Van Der Laan sequences as follows.

Corollary 10 *i) Let $a = b = c = 1$ in Theorem 9. Then, the determinant of the matrix $C(P) = \text{circ}(P_1, P_2, \dots, P_n)$ is written by*

$$\det(C(P)) = \frac{(P_n - 1)^n (1 - K^n - L^n + K^n L^n)}{(-1)^n (Q_{-n} - Q_n)},$$

$$\text{where } K = \frac{1 - P_{n+1} - \sqrt{(1 - P_{n+1})^2 - 4P_n P_{n-1} + 4P_{n-1}}}{2(P_n - 1)},$$

$$L = \frac{1 - P_{n+1} + \sqrt{(1 - P_{n+1})^2 - 4P_n P_{n-1} + 4P_{n-1}}}{2(P_n - 1)}.$$

ii) If we take $a = 3, b = 0, c = 2$ in Theorem 9. Then, the determinant of the matrix $C(Q) = \text{circ}(Q_1, Q_2, \dots, Q_n)$ is written by

$$\det(C(Q)) = \frac{(Q_n - 3)^n (1 - K^n - L^n + K^n L^n)}{(-1)^n (Q_{-n} - Q_n)},$$

where $K = \frac{-Q_{n+1} - \sqrt{Q_{n+1}^2 - 4Q_n Q_{n-1} - 4Q_n + 12Q_{n-1} + 12}}{2(Q_n - 3)}$ and

$$L = \frac{-Q_{n+1} + \sqrt{Q_{n+1}^2 - 4Q_n Q_{n-1} - 4Q_n + 12Q_{n-1} + 12}}{2(Q_n - 3)}.$$

iii) Let $a = c = 0, b = 1$ in Theorem 9. Then, the determinant of the matrix $C(R) = \text{circ}(R_1, R_2, \dots, R_n)$ is written by

$$\det(C(R)) = \frac{R_n^n (1 - K^n - L^n + K^n L^n)}{(-1)^n (Q_{-n} - Q_n)},$$

where $K = \frac{1 - R_{n+1} - \sqrt{(1 - R_{n+1})^2 - 4R_n R_{n-1}}}{2R_n},$

$$L = \frac{1 - R_{n+1} + \sqrt{(1 - R_{n+1})^2 - 4R_n R_{n-1}}}{2R_n}.$$

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